Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 51-32 (2018) 799–804

On the value of differential game with asymmetric control constraints

Vladimir Turetsky * Shmuel Y. Hayoun ** Tal Shima *** Alexander Tarasyev ****

* Ort Braude College of Engineering, Karmiel, Israel (e-mail: turetsky1@braude.ac.il)
** RAFAEL Advanced Defense Systems, Haifa, Israel (e-mail: shmuelyo@rafael.co.il)
*** Technion – Israel Institute of Technology, Haifa, Israel (e-mail: tal.shima@technion.ac.il)
**** Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, Ekaterinburg, Russia (e-mail: tam@imm.uran.ru)

Abstract: A differential game with asymmetric constraints on the players' controls and an asymmetric cost functional is considered. In this game hard geometric constraints are imposed on the maximizer, whereas the minimizer is soft-constrained by including the control effort term into the cost functional. The sufficient condition is derived, subject to which the program maximin is the game value. In the proof, it is shown that the program maximin is the generalized solution of the Hamilton-Jacobi-Bellman partial differential equation. Examples are presented.

© 2018, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.

Keywords: differential game, game value, program maximin, Isaacs equation.

1. INTRODUCTION

Differential game is a widely acceptable model for processes controlled by more than one agent in the conditions of conflict and uncertainty. This field of applied mathematics was pioneered by Isaacs (1954 – 1955, 1965). Its cornerstones were put in place by Fleming (1957); Berkovitz and Fleming (1957); Fleming (1961); Berkovitz (1964); Pontryagin (1966); Krasovskii and Subbotin (1988). Since then, the theory and applications of differential games demonstrated fruitful and extensive development. This concerns a variety of types of differential games (some dichotomic pairs are linear/nonlinear, determenistic/stochastic, continuous/discrete, antagonistic/cooperative, two-person/multiple-person, zero-sum/nonzero-sum, finite/infinite horizon, etc.), as well as their formalizations. In this paper, we consider a deterministic linear two-person finite-horizon zero-sum differential game.

The central question of the theory of such differential games is the existence and calculation of the game value, i.e., the equal guaranteed result of both players. Knowing the value function allows constructing optimal strategies by, e.g., the extremal shift procedure of Krasovskii and Subbotin (1988). If the game value is sufficiently smooth it can result in continuous feedback strategies, as in linear-quadratic differential games (Bryson and Ho (1975)). Alternatively, optimal strategies can be constructed in a discrete scheme (Krasovskii and Subbotin (1988); Krasovskii and Krasovskii (1994)).

Despite the maturity of the science, the game solution still represents a challenge. If a smooth function is the game value, it should satisfy the Isaacs partial differential equation Isaacs (1965). However, in many cases, the game value is not smooth. This fact inspired various generalizations of the Isaacs equation, in particular, the minimax solution of Subbotin (1994) and differential inequalities for conjugate derivatives (Subbotin and Tarasyev (1985)). In these works it was proved that, subject to some assumptions, if the function satisfies the Isaacs equation in the generalized sense, it turns out to be the game value.

In this paper, a differential game with asymmetric control constraints is considered. In this game, the maximizer's control is bounded by a hard constraint, whereas the minimizer's control is only soft-constrained by including a quadratic integral control effort term into the cost functional, in addition to a quadratic state term. Such a game (see Hayoun et al. (2016)) models a special type of an interception problem where the missile (pursuer) has a very large lateral acceleration capability, but no thrust capacity, whereas the target (evader) is a manned aircraft with a limited lateral acceleration, but with thrust capacity. A similar asymmetric differential game was considered by Gutman and Leitmann (1975). The sufficient condition is derived guaranteeing that a program maximin is the game value. It is proved by a straightforward check of differential inequalities of Subbotin and Tarasyev (1985). Examples are presented where the condition is satisfied and not satisfied.

2. PROBLEM STATEMENT

Consider a linear system

$$\dot{x} = A(t)x + b(t)u + c(t)v, \; x(t_*) = x_*, \; t \in [t_*, t_f], \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, u and v are the scalar players' controls, $u(\cdot), v(\cdot) \in L_2[t_*, t_f]$; t_f is the prescribed

2405-8963 © 2018, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2018.11.463

final time moment; $t_* \in [0, t_f)$ is the initial time moment; the matrix function $A(t)$ and the vector functions $b(t)$, $c(t)$ are continuous. It is assumed that the control $v(\cdot)$ satisfies the hard constraint given by $M > 0$. Thus,

$$u(\cdot) \in U = L_2[t_*, t_f], \tag{2}$$

$$v(\cdot) \in V = \{v(\cdot) \in L_2[t_*, t_f] : \ |v(t)| \le M, \quad t \in [t_*, t_f]\}. \tag{3}$$

The cost functional is

$$J_x = x_1^2(t_f) + \alpha \int_{t_*}^{t_f} u^2(t)dt \to \min_{u(\cdot)\in U} \max_{v(\cdot)\in V}, \tag{4}$$

where $\alpha > 0$ is the first player's control penalty coefficient.

Let us employ the terminal projection transformation (Bryson and Ho (1975))

$$z = D\Phi(t_f, t)x, \tag{5}$$

where $D = [1, 0, ..., 0]$ and $\Phi(t, t_0)$ is the fundamental matrix of a homogeneous system $\dot{x} = A(t)x$. By (5), the system (1) is converted into a scalar one:

$$\dot{z} = h_1(t)u + h_2(t)v, \tag{6}$$

where $h_1(t) = D\Phi(t_f, t)b(t)$, $h_2(t) = D\Phi(t_f, t)c(t)$. The cost functional becomes

$$J_z = z^2(t_f) + \alpha \int_{t_*}^{t_f} u^2(t)dt \to \min_{u(\cdot)\in U} \max_{v(\cdot)\in V}. \tag{7}$$

The objective of this paper is calculating the value of the differential game (6) – (7) with constraints (2) – (3).

3. GAME VALUE

3.1 Main result

Let us define the program maximin

$$\rho(t_*, z_*) = \max_{v(\cdot)\in V} \min_{u(\cdot)\in U} J_z. \tag{8}$$

Theorem 1. If for $t_* \in [0, t_f]$,

$$F(t_*) \triangleq$$
$$h_2(t_*)\left(\alpha + \int_{t_*}^{t_f} h_1^2(t)dt\right) - h_1^2(t_*)\int_{t_*}^{t_f} h_2(t)dt \ge 0, \tag{9}$$

then the program maximin (8) is the value of the differential game (6) – (7) with constraints (2) – (3)

The proof consists of several stages.

3.2 Calculation of the program maximin

It is directly observed that

$$z^2(t_f) = \max_{\lambda\in R}\left[\lambda z(t_f) - \frac{1}{4}\lambda^2\right]. \tag{10}$$

Due to (10) and

$$z(t_f) = z_* + \int_{t_*}^{t_f} h_1(t)u(t)dt + \int_{t_*}^{t_f} h_2(t)v(t)dt, \tag{11}$$

the program maximin is represented as

$$\rho(t_*, z_*) = \max_{v(\cdot)\in V} \min_{u(\cdot)\in U} \max_{\lambda\in R} \chi(u(\cdot), v(\cdot), \lambda), \tag{12}$$

where

$$\chi(u(\cdot), v(\cdot), \lambda) \triangleq$$
$$\lambda z_* + \lambda \int_{t_*}^{t_f} h_1(t)u(t)dt + \lambda \int_{t_*}^{t_f} h_2(t)v(t)dt +$$
$$\alpha \int_{t_*}^{t_f} u^2(t)dt - \frac{1}{4}\lambda^2. \tag{13}$$

Proposition 1. For any fixed admissible $v(t)$,

$$\min_{u(\cdot)} \max_{\lambda\in R} \chi(u(\cdot), v(\cdot), \lambda) = \max_{\lambda\in R} \min_{u(\cdot)} \chi(u(\cdot), v(\cdot), \lambda). \tag{14}$$

The proposition is proved by a straightforward calculation of maximizing and minimizing elements.

Due to Proposition 1, and to the fact that maximization operators commute, the program maximin is rewritten as

$$\rho(t_*, z_*) = \max_{\lambda\in R} \max_{v(\cdot)\in V} \min_{u(\cdot)\in U} \chi(u(\cdot), v(\cdot), \lambda). \tag{15}$$

For a fixed $\lambda \in R$, the maximizing function is

$$v^\lambda = M \operatorname{sign} \lambda. \tag{16}$$

The minimizing function is

$$u^\lambda = -\frac{h_1(t)\lambda}{2\alpha}. \tag{17}$$

By some algebra and by changing (t_*, z_*) with (t, z),

$$\rho(t, z) = \max_{\lambda\in R} \chi(u^\lambda(\cdot), v^\lambda(\cdot), \lambda) =$$
$$\begin{cases} \dfrac{\alpha\left(z + \operatorname{sign}(z)M\displaystyle\int_t^{t_f} h_2(\tau)d\tau\right)^2}{\alpha + \displaystyle\int_t^{t_f} h_1^2(\tau)d\tau}, & z \ne 0, \\[2ex] \dfrac{\alpha M^2\left(\displaystyle\int_t^{t_f} h_2(\tau)d\tau\right)^2}{\alpha + \displaystyle\int_t^{t_f} h_1^2(\tau)d\tau}, & z = 0. \end{cases} \tag{18}$$

3.3 Properties of the program maximin

First, it is seen that the function $\rho(t, z)$, given by (18), satisfies the boundary condition

$$\rho(t_f, z) = z^2. \tag{19}$$

The continuity and differentiability properties of $\rho(t, z)$ are formulated in the following proposition.

Proposition 2. The function $\rho(t, z)$ and its time derivative $\dfrac{\partial\rho}{\partial t}$ are continuous with respect to z. The derivative $\dfrac{\partial\rho}{\partial z}$ is continuous for $z > 0$ and $z < 0$, and has a finite discontinuity for $z = 0$.

Proof. It is obtained directly from (18) that

$$\lim_{z\to 0+} \rho(t, z) = \lim_{z\to 0-} \rho(t, z) =$$

$$\frac{\alpha M^2 \left(\int_t^{t_f} h_2(\tau)d\tau\right)^2}{\alpha + \int_t^{t_f} h_1^2(\tau)d\tau} = \rho(t,0). \tag{20}$$

Let us calculate the time derivative. For $z \neq 0$:

$$\frac{\partial \rho}{\partial t} = \frac{\alpha h_1^2(t)\left(z + \mathrm{sign}(z) M \int_t^{t_f} h_2(\tau)d\tau\right)^2}{\left(\alpha + \int_t^{t_f} h_1^2(\tau)d\tau\right)^2} - \mathrm{sign}(z)\frac{2\alpha M h_2(t)\left(z + \mathrm{sign}(z) M \int_t^{t_f} h_2(\tau)d\tau\right)}{\alpha + \int_t^{t_f} h_1^2(\tau)d\tau}. \tag{21}$$

It is seen that

$$\lim_{z\to 0+}\frac{\partial \rho(t,z)}{\partial t} = \lim_{z\to 0-}\frac{\partial \rho(t,z)}{\partial t} = \frac{\alpha h_1^2(t) M^2 \left(\int_t^{t_f} h_2(\tau)d\tau\right)^2}{\left(\alpha + \int_t^{t_f} h_1^2(\tau)d\tau\right)^2} - \frac{2\alpha h_2(t) M^2 \int_t^{t_f} h_2(\tau)d\tau}{\alpha + \int_t^{t_f} h_1^2(\tau)d\tau} = \frac{\partial \rho(t,0)}{\partial t}. \tag{22}$$

Let us calculate the derivative w.r.t. z. For $z \neq 0$,

$$\frac{\partial \rho}{\partial z} = \frac{2\alpha\left(z + \mathrm{sign}(z) M \int_t^{t_f} h_2(\tau)d\tau\right)}{\alpha + \int_t^{t_f} h_1^2(\tau)d\tau}. \tag{23}$$

It is seen that

$$\lim_{z\to 0+}\frac{\partial \rho(t,z)}{\partial z} = \frac{2\alpha M \int_t^{t_f} h_2(\tau)d\tau}{\alpha + \int_t^{t_f} h_1^2(\tau)d\tau} = \frac{\partial \rho(t,0+)}{\partial z}, \tag{24}$$

$$\lim_{z\to 0-}\frac{\partial \rho(t,z)}{\partial z} = -\frac{2\alpha M \int_t^{t_f} h_2(\tau)d\tau}{\alpha + \int_t^{t_f} h_1^2(\tau)d\tau} = \frac{\partial \rho(t,0-)}{\partial z}, \tag{25}$$

and $\frac{\partial \rho}{\partial z}$ has a finite discontinuity for $z = 0$. □

3.4 Isaacs equation

Let us write down the Isaacs partial differential equation (Isaacs (1965)) for the value $V(t,z)$ of the game (6) – (7), (2) – (3):

$$\frac{\partial V}{\partial t} + H\left(t, z, \frac{\partial V}{\partial z}\right) = 0, \tag{26}$$

where the Hamiltonian H is

$$H(t,z,s) = \min_{u\in R}\max_{|v|\le M}\left[s(h_1(t)u + h_2(t)v) + \alpha u^2\right] = \min_{u\in R}\left[s b_p(t) u + \alpha u^2\right] + \max_{|v|\le M}\left[s b_e(t) v\right]. \tag{27}$$

The minimizing and the maximizing elements are

$$\tilde{u} = -\frac{h_1(t)s}{2\alpha}, \tag{28}$$

and

$$\tilde{v} = M\,\mathrm{sign}(s), \tag{29}$$

respectively. Due to (27) – (29),

$$H(t,z,s) = -\frac{h_1^2(t)s^2}{4\alpha} + M h_2(t)|s|. \tag{30}$$

Thus, the equation (26) becomes

$$\frac{\partial V}{\partial t} + \left(-\frac{h_1^2(t)}{4\alpha}\left(\frac{\partial V}{\partial z}\right)^2 + M h_2(t)\left|\frac{\partial V}{\partial z}\right|\right) = 0. \tag{31}$$

Proposition 3. The program maximin (18) satisfies the Isaacs - Bellman equation (31) for $z > 0$ and for $z < 0$.

Proof. Note that due to (23),

$$\frac{\partial \rho(t,z)}{\partial z} > 0 \text{ for } z > 0 \text{ and } \frac{\partial \rho(t,z)}{\partial z} < 0 \text{ for } z < 0. \tag{32}$$

Therefore, by virtue of (21), for $z > 0$,

$$\frac{\partial \rho}{\partial t} - \frac{h_1^2(t)}{4\alpha}\left(\frac{\partial \rho}{\partial z}\right)^2 + M h_2(t)\left|\frac{\partial \rho}{\partial z}\right| =$$

$$\frac{\partial \rho}{\partial t} - \frac{h_1^2(t)}{4\alpha}\left(\frac{\partial \rho}{\partial z}\right)^2 + M h_2(t)\frac{\partial \rho}{\partial z} =$$

$$\frac{\alpha h_1^2(t)\left(z + M \int_t^{t_f} h_2(\tau)d\tau\right)^2}{\left(\alpha + \int_t^{t_f} h_1^2(\tau)d\tau\right)^2} - \frac{2\alpha M h_2(t)\left(z + M \int_t^{t_f} h_2(\tau)d\tau\right)}{\alpha + \int_t^{t_f} h_1^2(\tau)d\tau} - \frac{h_1^2(t)}{4\alpha}\cdot\frac{4\alpha^2\left(z + M \int_t^{t_f} h_2(\tau)d\tau\right)^2}{\left(\alpha + \int_t^{t_f} h_1^2(\tau)d\tau\right)^2} +$$

$$\frac{2\alpha M h_2(t)\left(z+M\int\limits_t^{t_f} h_2(\tau)d\tau\right)}{\alpha+\int\limits_t^{t_f} h_1^2(\tau)d\tau}=0. \tag{33}$$

Similarly, it is shown that $\rho(t,z)$ satisfies the Isaacs-Bellman equation for $z<0$. □

3.5 Differential inequalities

In order to prove that $\rho(t,z)$ is the game value for $z=0$ (where $\frac{\partial\rho}{\partial z}$ has discontinuity), we show that $\rho(t,z)$ satisfies the differential inequalities, generalizing the Isaacs equation (Subbotin and Tarasyev (1985)). The lower and the upper Dini directional derivatives of the function $V(t,z)$ are defined as

$$\partial_- V(t,z)|(1,h)=\liminf_{\delta\to 0}\frac{V(t+\delta,z+\delta h)-V(t,z)}{\delta}, \tag{34}$$

and

$$\partial_+ V(t,z)|(1,h)=\limsup_{\delta\to 0}\frac{V(t+\delta,z+\delta h)-V(t,z)}{\delta}, \tag{35}$$

respectively. The lower and the upper conjugate derivatives are defined as

$$D^*V(t,z)|s=\sup_{h\in R}\left[sh-\partial_- V(t,z)|(1,h)\right], \tag{36}$$

and

$$D_*V(t,z)|s=\inf_{h\in R}\left[sh-\partial_+ V(t,z)|(1,h)\right], \tag{37}$$

The differential inequalities, generalizing the Isaacs equation, are

$$D^*V(t,z)|s\ge H(t,z,s), \tag{38}$$

$$D_*V(t,z)|s\le H(t,z,s), \tag{39}$$

where $H(t,z,s)$ is the Hamiltonian (30). The inequalities (38) – (39) constitute the sufficient condition for a function $V(t,z)$ to be the game value (Subbotin and Tarasyev (1985)). In the points where $V(t,z)$ is differentiable, they reduce to the Isaacs equation (31).

Proposition 4. If the condition (9) holds, then the program maximin (18) satisfies the differential inequalities (38) – (39) for $z=0$.

Proof. Omitting some algebra,

$$D^*\rho(t,0)|s=\begin{cases}-\dfrac{\partial\rho(t,0)}{\partial t}, & |s|\le\kappa(t),\\[2ex] \infty, & s<-\kappa(t) \text{ or } s>\kappa(t),\end{cases} \tag{40}$$

where

$$\kappa(t)\triangleq\frac{2\alpha M\int\limits_t^{t_f} h_2(\tau)d\tau}{\alpha+\int\limits_t^{t_f} h_1^2(\tau)d\tau}. \tag{41}$$

Thus, for $s\notin[-\kappa(t),\kappa(t)]$, $\rho(t,z)$ satisfies the differential inequality (38) at $(t,0)$. Let $|s|\le\kappa(t)$. Let us show that in this case (38) is also valid for $\rho(t,z)$ at $(t,0)$, i.e.,

$$-\frac{\partial\rho(t,0)}{\partial t}\ge H(t,z,s),\quad |s|\le\kappa(t). \tag{42}$$

Due to (22) and (30), this means that

$$\frac{2\alpha h_2(t)M^2\int\limits_t^{t_f} h_2(\tau)d\tau}{\alpha+\int\limits_t^{t_f} h_1^2(\tau)d\tau}-\frac{\alpha h_1^2(t)M^2\left(\int\limits_t^{t_f} h_2(\tau)d\tau\right)^2}{\left(\alpha+\int\limits_t^{t_f} h_1^2(\tau)d\tau\right)^2}\ge$$

$$Mh_2(t)|s|-\frac{h_1^2(t)s^2}{4\alpha}, \tag{43}$$

for $|s|\le\kappa(t)$. By (41), the inequality (43) can be rewritten as:

$$Mh_2(t)\kappa(t)-\frac{h_1^2(t)\kappa^2(t)}{4\alpha}\ge Mh_2(t)|s|-\frac{h_1^2(t)s^2}{4\alpha}. \tag{44}$$

The function $H(t,z,s)$ in the right-hand side of (44) is even and admits its global maximum for

$$s=\pm s^*=\pm\frac{2\alpha M h_2(t)}{h_1^2(t)}. \tag{45}$$

Now, the inequality (44) holds if

$$s^*\ge\kappa(t). \tag{46}$$

Due to (41) and (45), this leads to the condition

$$\frac{2\alpha M h_2(t)}{h_1^2(t)}\ge\frac{2\alpha M\int\limits_t^{t_f} h_2(\tau)d\tau}{\alpha+\int\limits_t^{t_f} h_1^2(\tau)d\tau}, \tag{47}$$

which yields (9) for $t_*=t$.

The second differential inequality (39) is treated in the same fashion. This completes the proof of the proposition. □

Propositions 3 – 4, along with the boundary condition (19), prove Theorem 1.

If the condition (9) holds, then, due to (28) – (29) and (23), the optimal strategies for $z>0$ and for $z<0$ are

$$u^0(t,z)=-\frac{h_1(t)\left(z+\mathrm{sign}(z)M\int\limits_t^{t_f} h_2(\tau)d\tau\right)}{\alpha+\int\limits_t^{t_f} h_1^2(\tau)d\tau}, \tag{48}$$

$$v^0(t,z)=M\,\mathrm{sign}(z). \tag{49}$$

4. EXAMPLES

4.1 Scalar game with simple motions

In this example, $h_1(t)=h_2(t)\equiv 1$:

$$\dot z=u+v. \tag{50}$$

In this case, $F(t)\equiv\alpha$ where the function F is defined in (9). Thus, the condition (9) holds and the program maximin $\rho(t_*,z_*)$ is the value of the game for the system

(50) with the cost functional (7) and constraints (2) – (3). By substituting (48) – (49) into (50) and by solving the differential equation, the optimal trajectories are the straight lines

$$z^0(t) = \begin{cases} M\alpha + \dfrac{(z_* - M\alpha)(\alpha + t_f - t)}{\alpha + t_f - t_*}, & z_* > 0, \\ -M\alpha + \dfrac{(z_* + M\alpha)(\alpha + t_f - t)}{\alpha + t_f - t_*}, & z_* < 0. \end{cases} \quad (51)$$

These trajectories are symmetric w.r.t. the t-axis. For $z_* > 0$, they decrease for $z_* > M\alpha$ and increase for $0 < z_* < M\alpha$ (see Fig. 1 where the optimal trajectories are shown as solid lines).

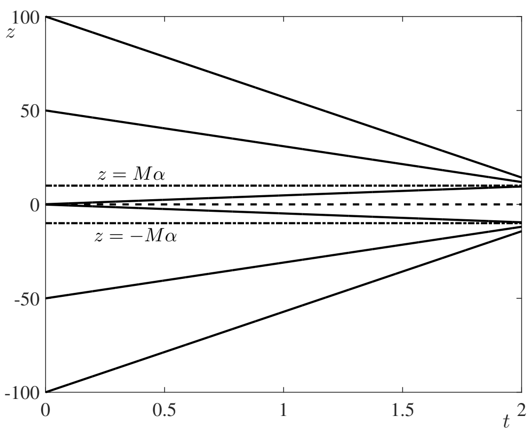

Fig. 1. Optimal trajectories for scalar game with simple motions: $t_f = 2$, $M = 100$, $\alpha = 0.1$

4.2 Pursuit-evasion game with first-order dynamics of the players

The game in this example models a planar engagement between two maneuvering vehicles (pursuer and evader) with first-order controller dynamics. Subject to the assumption that the heading angles and the line-of-sight angle are small, the engagement can be described (see, e.g., Turetsky and Shinar (2003)) by a linear system (1) where $x = (x_1, x_2, x_3, x_4)^T$, x_1 is the relative separation between the vehicles normal to the initial line-of-sight; x_2 is the relative velocity; x_3 and x_4 are the lateral accelerations, u and v are the lateral acceleration commands (controls) of the pursuer and the evader, respectively. The matrices are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1/\tau_p \\ 0 & 0 & -1/\tau_e & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1/\tau_p \\ 0 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\tau_e \end{bmatrix}, \quad (52)$$

where τ_p and τ_e are the time constants of the pursuer and the evader, respectively. In the scalarized system (6),

$$h_1(t) = -h(t_f - t, \tau_p), \quad h_2(t) = h(t_f - t, \tau_e), \quad (53)$$

where

$$h(t, \tau) \triangleq \tau\left(\exp(-t/\tau) + t/\tau - 1\right). \quad (54)$$

In this example, there exists a time moment $t^* \in (0, t_f)$ such that the condition (9) is valid for $t \in [t^*, t_f]$ (see the graph of the function $F(t)$ in Fig. 2 for $t_f = 2$, $M = 100, \tau_p = 0.2, \tau_e = 0.1, \alpha = 0.1$: for these parameters, $t^* = 1.04$).

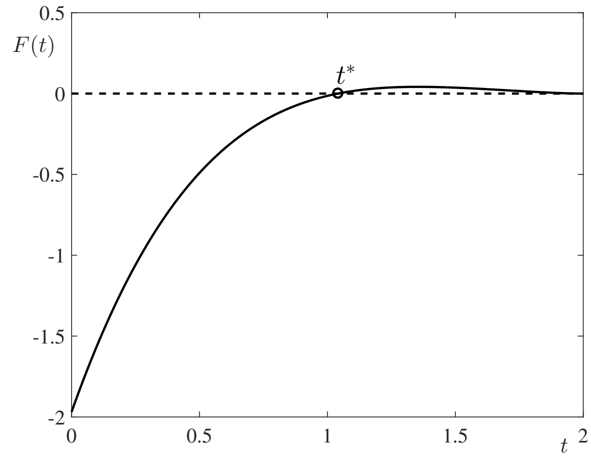

Fig. 2. Function $F(t)$ for first-order pursuer and evader

For $z = 0$, $t \in [0, t^*)$, the program maximin is not the game value. In this case, the game space $S = \{(t, z) : t \in [0, t_f], z \in R\}$ is decomposed into two regions by using two trajectories, generated by the pair (48) – (49), symmetric w.r.t. to the axis $z = 0$ and tangent to it at $t = t^*$. These trajectories are $z = \pm z^*(t)$ where

$$z^*(t) = M \left[\frac{\alpha + \int_t^{t_f} h_1^2(\tau)d\tau}{\alpha + \int_{t^*}^{t_f} h_1^2(\tau)d\tau} \int_{t^*}^{t_f} h_2(\tau)d\tau - \int_t^{t_f} h_2(\tau)d\tau \right]. \quad (55)$$

In Fig. 3, they are shown by bold lines, solid for $t \in [0, t^*)$ and dashed for $t \in [t^*, t_f]$. The regular region

$$D_1 = \{(t, z) : \ t \in [0, t^*], |z| \geq z^*(t) \wedge t \in [t^*, t_f], z \in R\}, \quad (56)$$

is completely covered by the trajectories generated by the optimal strategies (48) – (49): they are shown in Fig. 3 by thin solid lines. The singular region is $D_0 = S \backslash D_1$.

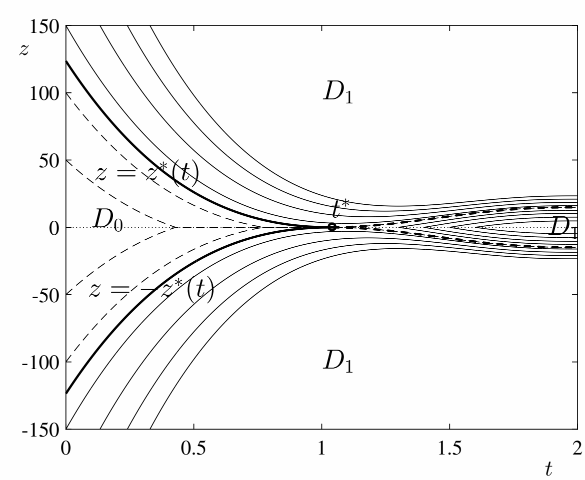

Fig. 3. Game space decomposition

Proposition 5. Any trajectory, generated by (48) – (49) from $(t_*, z_*) \in D_0$, $z_* \neq 0$, approaches the $z = 0$ axis monotonically and reaches it at some moment $\bar{t} \in (t_*, t^*)$. For $t \in [\bar{t}, t^*]$, it is a sliding mode trajectory along $z = 0$.

Proof. Let $z_* > 0$. By substituting (48) – (49) into (6), one obtains the differential equation

$$\dot{z} = \frac{-h_1^2(t)z + MF(t)}{\alpha + \int_t^{t_f} h_1^2(\tau)d\tau}. \tag{57}$$

Since $F(t) < 0$ for $t \in [0, t^*)$,

$$\dot{z} < 0, \quad t \in [0, t^*), \tag{58}$$

and all such trajectories decrease on $[t_*, t^*]$. By direct integration of (57),

$$z(t) = \frac{z_* + M\int_{t_*}^{t_f} h_2(\tau)d\tau}{\alpha + \int_{t_*}^{t_f} h_1^2(\tau)d\tau}\left(\alpha + \int_t^{t_f} h_1^2(\tau)d\tau\right) - M\int_t^{t_f} h_2(\tau)d\tau. \tag{59}$$

Thus, if two trajectories $z_1(t)$ and $z_2(t)$ emanate from (t_*, z_{*1}) and (t_*, z_{*2}), $z_{*1} < z_{*2}$, then $z_1(t) < z_2(t)$, i.e., the trajectory emanating from D_0 cannot intersect its boundary $z = z^*(t)$ and reaches $z = 0$ at some moment $\bar{t} \in (t_*, t^*)$.

Similarly, for $z_* < 0$: $\dot{z} > 0$ for $t \in [0, t^*)$, which, along with (58) yields $z\dot{z} < 0$ and guarantees the sliding mode along $z = 0$. □

Such trajectories are shown in Fig. 3 by thin dashed lines. Once the trajectory reaches the axis $z = 0$ and starts the sliding mode motion, it ceases being optimal, because the program maximin is not a game value for $z = 0$, $t \in [0, t^*)$.

5. CONCLUSIONS

For the differential game with a hard constraint on the maximizer's control and with a soft constraint on the minimizer's control, the sufficient condition is derived, guaranteeing that the program maximin is the game value. This fact is proved by a dual calculation of the program maximin and by using the differential inequalities generalizing the Isaacs equation.

Two examples are presented. In the scalar game with simple motions, this condition is satisfied. In the scalarizable pursuit-evasion game with first-order dynamics of the players the condition is violated at the interval of the state variable axis. This yields the state space decomposition into the regular and the singular regions. In the regular region the program maximin is the game value and the optimal strategies are constructed based on its space derivative. In the singular region, all candidate optimal trajectories approach the horizontal axis, after which the sliding mode begins and these trajectories cease being optimal. The problem of constructing the game value in the singular zone is the topic of future research.

REFERENCES

Berkovitz, L.D. (1964). A variational approach to differential games. In M. Dresher, L.S. Shapley, and A. Tucker (eds.), *Advances in Game Theory*, volume 52 of *Annals of Mathematics Studies*, 127 – 174. Princeton University Press, Princeton, N.J.

Berkovitz, L.D. and Fleming, W.H. (1957). On differential games with integral payoff. In M. Dresher, A. Tucker, and P. Wolfe (eds.), *Contributions to the Theory of Games III*, volume 39 of *Annals of Mathematics Studies*, 413 – 435. Princeton University Press, Princeton, N.J.

Bryson, A. and Ho, Y. (1975). *Applied Optimal Control*. Hemisphere, New York, NY.

Fleming, W.H. (1957). A note on differential games of prescribed duration. In M. Dresher, A. Tucker, and P. Wolfe (eds.), *Contributions to the Theory of Games III*, volume 39 of *Annals of Mathematics Studies*, 407 – 412. Princeton University Press, Princeton,N.J.

Fleming, W.H. (1961). The convergence problem for differential games. *Journal of Mathematical Analysis and Applications*, 3, 102 – 116.

Gutman, S. and Leitmann, G. (1975). On a class of differential games. *Journal of Optimization Theory and Applications*, 17(5 – 6), 511 – 522.

Hayoun, S.Y., Weiss, M., and Shima, T. (2016). A mixed L_2/L_∞ differential game approach to pursuit-evasion guidance. *IEEE Transactions on Aerospace and Electronic Systems*, 52(6), 2775 – 2788.

Isaacs, R. (1954 – 1955). Differential Games I, II, III, IV. Reseasrch Mermoranda RM-1391, RM-1399, RM-1411, RM-1486, The RAND Corporation.

Isaacs, R. (1965). *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*. John Wiley and Sons, Inc. Dover Publications, Inc., New York, N.Y.

Krasovskii, A.N. and Krasovskii, N.N. (1994). *Control Under Lack of Information*. Systems & Control: Foundations & Applications. Birkhäuser, Boston, Basel, Berlin.

Krasovskii, N. and Subbotin, A. (1988). *Game-Theoretical Control Problems*. Springer Verlag, New York, N.Y.

Pontryagin, L.S. (1966). On the theory of differential games. *Russian Mathematical Surveys*, 21(4), 193 – 246.

Subbotin, A.I. and Tarasyev, A.M. (1985). Conjugate derivatives of the value function of a differential game. *Soviet Mathematics Doklady*, 32, 162 – 166.

Subbotin, A.I. (1994). *Generalized solutions of first-order PDEs: the dynamical optimization perspective*. Systems & Control: Foundations & Applications. Birkhäuser, Boston, Basel, Berlin.

Turetsky, V. and Shinar, J. (2003). Missile guidance laws based on pursuit-evasion game formulations. *Automatica*, 39(4), 607 – 618.